Input design and online system identification based on Poisson moment functions for system outputs with quantization noise

Simon Mayr, Gernot Grabmair and Johann Reger

***Abstract*— We study optimal input design and bias-compensating parameter estimation methods for continuous-time models applied on a mechanical laboratory experiment. Within this task we compare two online estimation methods that are based on Poisson moment functions with focus on quantized system outputs due to an angular encoder: The standard recursive least-squares (RLS) approach and a bias-compensating recursive least-squares (BCRLS) approach. The rationale is to achieve acceptable estimation results in the presence of white noise, caused by low-budget encoders with low resolution. The input design and parameter estimation approaches are assessed and compared, experimentally, resorting to measurements taken from a laboratory cart system.**

I. Introduction

For the controller design it is crucial that the system behavior in simulation outlines the behavior of the real plant. In addition to the plant modeling itself, the accurate identification of system parameters is central. For process and failure monitoring tasks it is necessary to detect parameter changes immediately to avoid malfunctions even when measurements are corrupted by noise. In this regard, identification requires a sufficient excitation of the plant. However, choosing valid plant input signals with sufficient excitation while keeping the wear of the plant low is demanding in many industrial projects. Towards this end, the automatic generation of such plant inputs by solving an optimization problem is appealing. It is already common practice to generate trajectories for mechanical systems by optimization techniques. Especially for mechanical plants, customary approaches include time optimal, energy optimal, and other motion profiles. The approaches allow to integrate machine constraints easily. For decreasing mechanical wear such profiles often need to provide, among others, smooth (rest-to-rest) movements in accordance with restrictions in jerk, acceleration, and velocity. On the other hand, smooth trajectories tend to contain insufficient excitation information for identification. Therefore, metrics are necessary for capturing the information content concerning the unknown parameters in the optimal control problem, e.g. see [5], [6].

In our approach we adhere to identify continuous-time models because these models may grant further insight into the physical system properties. Abundant literature deals with these topics, both in discrete and in continuous time. For parameter estimation of continuous-time models of dynamical systems many approaches are based on time derivatives of some measured signals, which are not available in general. Time-derivatives of the signals often are approximated, in other approaches they may also be eliminated, provided that particular assumptions are satisfied. Section III of our paper gives a brief overview of customary identification approaches, see [2], [8], [9], [11], [12], [13], [15].

TABLE I
Explanation of abbreviations.

Abbreviation	Description
PMF	Poisson moment functions
DOE	design of experiments
(R)LS	(recursive) least-squares
BC(R)LS	bias-compensated (recursive) least-squares

Finally, estimates for the unknown parameters are obtained by solving the respective set of equations with simple least-squares. It is well-known that the LS estimator is unbiased in presence of white noise with zero mean. In general, however, the assumed serially uncorrelated output noise will be transformed into arbitrary colored noise due to the elimination of the time-derivatives wrt. the measured signals. Employing Poisson moment functions for identification, the LS estimator may still show a good usability for small measurement noise due to the inherent low-pass characteristic of the approach. Nevertheless, larger measurement noises usually lead to noticeably biased estimation results. For reducing bias, compensation approaches such as instrumental variables or variations of bias-compensating least-squares (BCLS) have been developed, see [12], [13], [14].

Against this background, the main contribution of the underlying paper is an application-oriented study of an online recursive bias-compensating least-squares (BCRLS) approach for continuous-time systems with quantization noise caused by encoders, for example. The approach is based on Poisson moment functions (PMF) [13] and applied on a laboratory experiment for obtaining the parameter estimates in real-time, suitable for failure monitoring. We compare the identification results of the RLS algorithm and the BCRLS algorithm with focus on low budget applications with low encoder resolution, especially interesting for industrial applications, and online parameter monitoring with standard input signals and optimal input signals, i.e. design of experiments (DOE). Our results clearly show that the BCRLS is superior to the RLS, particularly for low resolution encoders. We further reveal that dedicated DOE input signals

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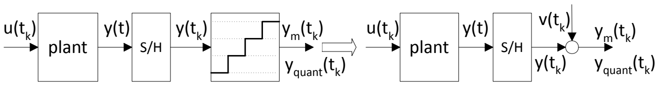

Fig. 1. Modeling quantization error as additive white noise.

may further improve the results.

Note that although we focus on the practical application of bias-compensating identification approach, the basic ideas of the employed identification method will be exposed in detail for improving the reading comprehension, see III-B.

The paper is organized as follows: Subsequent to this introduction, Section II briefly discusses the optimization problem for optimal input design and trajectory generation. Section III gives a short overview over advantages and disadvantages of continuous-time identification. Afterwards, an online bias-compensating least-squares algorithm based on PMF for linear SISO systems is presented. Finally, the optimal input design and the parameter estimation methods are assessed referring to a laboratory experiment in Section IV. Conclusions are drawn in Section V.

II. Optimal input design

We consider strictly proper observable and controllable SISO-LTI systems of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\left(\mathbf{p}\right)\mathbf{x} + \mathbf{b}\left(\mathbf{p}\right)u \\ y &= \mathbf{c}^{\mathrm{T}}\left(\mathbf{p}\right)\mathbf{x}\end{aligned} \tag{1}$$

with state $\mathbf{x}(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, and constant (unknown) parameter vector $\mathbf{p} \in \mathbb{R}^{n_p}$, equivalent to the input-output representation in (6) that we use for parameter estimation. The output of the continuous-time system is measured at various discrete instances of time, t_k, $k = 1, 2, \ldots, N$, where N is the total number of samples. The samples

$$y_m(k) = y_{quant}(k) \approx y(k) + v(k), \quad 1 \leq k \leq N \tag{2}$$

are quantized by

$$y_{quant}(y(k)) = \Delta \left\lfloor \frac{y(k)}{\Delta} \right\rfloor \tag{3}$$

with quantization step-size Δ, where $\lfloor \cdot \rfloor$ denotes the floor (round down) operator. Assuming significant changes from sample to sample, it is common practice to approximate quantized system outputs by its pure sampled output $y(k)$ and an additive sequence of uncorrelated white noise $v(k)$ with constant and known variance σ^2, see [16], [17] and Fig. 1, respectively[1]. As a result, the deterministic nonlinear function is replaced with a stochastic linear one.

Remark: For DOE wrt. a more general class of systems the reader is referred to [6].

In order to ensure an input signal that is sufficiently "rich" for identification while also respecting plant limitations we combine the standard optimal control problem with an input design approach. The initial guess of unknown parameters $\hat{\mathbf{p}}_0 \approx \mathbf{p}$ is assumed to be chosen accurate enough for input design. For analyzing the information content of a measurement we resort to the Fisher information matrix $\mathbf{F}(u, \mathbf{p})$. This matrix indicates how much information a measurable system output contains about n_p unknown system parameters. As usual, the optimal input design task is assumed to be independent of the identification method used later on.

The Fisher information matrix is given by [5], [6]

$$\mathbf{F}(u, \mathbf{p}) = \sum_{k=1}^{N} \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k}^{\mathrm{T}} \left(\sigma^2\right)^{-1} \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k}$$
$$\text{where} \quad \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k} = \left(\left.\tfrac{\partial y}{\partial p_1}\right|_{\mathbf{p},k} \quad \cdots \quad \left.\tfrac{\partial y}{\partial p_{n_p}}\right|_{\mathbf{p},k} \right). \tag{4}$$

For optimization purposes this matrix has to be considered in a meaningful metric. Various real-valued functions have been suggested as a norm for the size of variance-covariance or the information matrix. The most common among them are the so-called A-criterion (trace of the matrix), D-criterion (determinant of the matrix) or E-criterion (eigenvalue of the matrix). More details and geometrical interpretations of these criteria may be found in [7].

Now, the input design problem can be formulated as optimal control problem

$$\begin{aligned}&\max_u \operatorname{trace}\left(\mathbf{F}(u, \mathbf{p})\right) \\ &\dot{\mathbf{x}} = \mathbf{A}\left(\mathbf{p}\right)\mathbf{x} + \mathbf{b}\left(\mathbf{p}\right)u,\ y = \mathbf{c}^{\mathrm{T}}\left(\mathbf{p}\right)\mathbf{x} \\ &\mathbf{x}\left(t_0\right) = \mathbf{x}_0,\ \mathbf{x}\left(t_e\right) = \mathbf{x}_e,\ u \in U,\ \mathbf{x} \in \mathbf{X} \\ &\dot{\mathbf{S}}_{\mathbf{x}} = \mathbf{A}\left(\mathbf{p}\right)\mathbf{S}_{\mathbf{x}} + \frac{\partial \mathbf{A}\left(\mathbf{p}\right)\mathbf{x} + \mathbf{b}\left(\mathbf{p}\right)u}{\partial \mathbf{p}},\ t > 0 \\ &\mathbf{S}_{\mathbf{x}}\left(0\right) = \frac{\partial \mathbf{x}_0}{\partial \mathbf{p}} \\ &\mathbf{s}_y = \mathbf{c}^{\mathrm{T}}\left(\mathbf{p}\right)\mathbf{S}_{\mathbf{x}} + \frac{\partial \mathbf{c}^{\mathrm{T}}\left(\mathbf{p}\right)\mathbf{x}}{\partial \mathbf{p}},\ t > 0\end{aligned} \tag{5}$$

with starting point t_0 and final time t_e.

The optimization problem with (forward) state sensitivities $\mathbf{S}_{\mathbf{x}}$ and output sensitivities $\mathbf{s}_y = \frac{\partial y}{\partial \mathbf{p}}$ may in practice be highly nonlinear, of course. Necessarily, we thus shall be satisfied with a suboptimal result.

Remark: As previously mentioned, for input design we assume a good initial guess $\hat{\mathbf{p}}_0$ of the plant parameters. Simulation experiments have shown that even poor initial guesses with parameter tolerances of about 20 percent turn out non-problematic concerning input constraints, see e.g. Fig. 2. In contrast, for state constraints and especially terminal conditions a sufficiently accurate estimate of the plant parameters is required. A possible remedy for this problem is to iterate over the tasks input design and parameter estimation.

III. Parameter Identification with PMF

In the nineties, large efforts have been made to develop the indirect approach for identification of continuous-time models, i.e. the identification of a discrete-time model and transformation into a continuous-time model. More recently,

[1]Note that the approximation of the quantizer in (3) is necessary for the mathematical derivations in Section III-A. Measurements and simulation experiments were performed with true quantizers, that is, optical encoders.

the identification of continuous-time models was experiencing a boost again. There are a number of advantages describing a system in a continuous-time model in favor of a discrete one. Among others, we find that

- continuous-time models may allow physical reasoning on the system properties,
- discretization needs a special handling at high sampling rates.

A detailed treatment and exposition of advantages and disadvantages may be found, e.g. in [9].

One of the most challenging problems in identification of continuous-time models is the handling of time-derivatives of input and output signals. There are basically two different routes to solve this problem. First, one may try to approximate or estimate the respective time-derivatives, e.g. see [2]. Another route is based on an exact system or signal transformation such that time-derivatives cannot arise. Most methods resort to an integration by parts. Typical approaches include the modulating functions [8], integration [4], Poisson moment function approaches [11] in combination with state-variable-filters [1], or frequency domain modulating functions [3], to name but a few.

A. *Identification with the PMF approach*

Equation (1) may be represented equivalently as

$$y^{(n)}(t) + \sum_{i=0}^{n-1} a_i y^{(i)}(t) = \sum_{j=0}^{m_d} b_j u^{(j)}(t) \tag{6}$$

with coefficients a_i, b_j, system order $n > m_d$ and parameter vector

$$\mathbf{p} = \left(a_0, \cdots, a_{n-1}, b_0, \cdots, b_{m_d}\right)^{\mathrm{T}} . \tag{7}$$

The integral transform

$$M_k\left\{y(t)\right\} = \int_0^t \frac{(t-\tau)^k}{k!} e^{-\lambda(t-\tau)} y(\tau)\,\mathrm{d}\tau \tag{8}$$

defines the k-th order Poisson moment of the signal $y(t)$ ($k = 0, 1, 2, \ldots$). Assuming that the support of y is restricted to $[0, \infty[$, this can be interpreted as a modulating function approach using convolution $g_f * y$ with impulse response g_f of a stable linear filter with transfer function[2]

$$G_F(s) = \frac{1}{(s+\lambda)^{k+1}} . \tag{9}$$

Integration by parts or Laplace transform shows the effect on the first derivatives[3]

$$M_k\left\{\dot{y}(t)\right\} = M_{k-1}\left\{y(t)\right\} - \lambda M_k\left\{y(t)\right\} - g_f(t)\,y(0) \tag{10}$$

where initially

$$M_0\left\{\dot{y}(t)\right\} = y(t) - \lambda M_0\left\{y(t)\right\} - g_f(t)\,y(0) . \tag{11}$$

This property is used again to get rid of all derivatives in (6) and to obtain a purely algebraic system of equations for parameter identification. The initial conditions are assumed to be zero or at least known, or will be neglected since their impact fades with time depending on the filter time constants. Whenever necessary, the initial conditions may also be estimated as further parameters.

Again, the filters (9) have to be discretized or approximated for the application on sampled measurement data. The plant input filters are discretized under the assumption that the inputs are generated in an exact manner by a zero-order hold. Any other filters are discretized using bilinear approximation, or Tustin's method (trapezoidal rule). More accurate approximations are possible, of course.

B. *Bias compensating least-squares estimation approach*

Even if we identify the continuous-time system in an exact manner, it is only necessary to evaluate the measured output at certain discrete points in time. Using y_m from (2) in (6) with (10), one arrives at

$$y_{PMF}^{(n)}(k) = \boldsymbol{\phi}^{\mathrm{T}}(k)\,\mathbf{p} + w(k) \tag{12}$$

with, PMF-induced, modified noise error w and regressor

$$\begin{aligned}\boldsymbol{\phi}^{\mathrm{T}}(k) =& \Big(-\bar{\boldsymbol{\Lambda}}_n \left(M_0\left\{y(k)\right\}, \cdots, M_n\left\{y(k)\right\}\right)^{\mathrm{T}} , \\ & \bar{\boldsymbol{\Lambda}}_{m_d}\left(M_{n-m_d}\left\{u(k)\right\}, \cdots, M_n\left\{u(k)\right\}\right)^{\mathrm{T}} \Big)\end{aligned} \tag{13}$$

and

$$\begin{aligned} y_{PMF}^{(n)}(k) =& M_n\left\{\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n}\right\}_{t_k} \\ =& \tilde{\boldsymbol{\Lambda}}_n \left(M_0\left\{y(k)\right\}, \cdots, M_n\left\{y(k)\right\}\right)^{\mathrm{T}} .\end{aligned} \tag{14}$$

For these expressions, the square coefficient matrix

$$\boldsymbol{\Lambda}_n(i,j) = \begin{cases} 0 & i > j \\ (-1)^{j-i} \frac{(n+1-i)!}{(n+1-j)!(j-i)!} \lambda^{j-i} \beta^{n+i-j} & i \leq j \end{cases} \tag{15}$$

may be partitioned according to

$$\boldsymbol{\Lambda}_n = \begin{pmatrix} \tilde{\boldsymbol{\Lambda}}_{1\times n} \\ \hline \bar{\boldsymbol{\Lambda}}_{(n-1)\times n} \end{pmatrix} \tag{16}$$

suitable for use in (13) and (14). Furthermore, λ is the filter time constant and β the filter gain, obtained by iterative application of (10).

If the system output $y(k)$ is disturbed by an additive zero-mean white noise $v(k)$ then the equation error $w(k)$ in (12) consists of colored noise, due to PMF filtering of $y_m(k)$ in (13). In light of this, we end up with a least-squares estimate based on N samples, i.e., the biased estimated parameter vector is

$$\begin{aligned}\hat{\mathbf{p}}_B(N) =& \hat{\mathbf{p}}(N) + \\ & \underbrace{\left(\frac{1}{N}\sum_{k=1}^{N} \boldsymbol{\phi}(k)\,\boldsymbol{\phi}^{\mathrm{T}}(k)\right)^{-1}}_{\mathbf{R}_{\phi\phi}^{-1}(N)} \underbrace{\left(\frac{1}{N}\sum_{k=1}^{N} \boldsymbol{\phi}(k)\,w(k)\right)}_{\mathbf{R}_{\phi w}(N)}\end{aligned} \tag{17}$$

[2]Additionally, the filter can be normalized to unity gain by λ^{k+1}.

[3]For a definition of Poisson moment functionals for distributions see [11].

where

$$\lim_{N\to\infty}\hat{\mathbf{p}}_B(N)=\mathbf{p}+\lim_{N\to\infty}\mathbf{R}_{\phi\phi}^{-1}\mathbf{R}_{\phi w}=\mathbf{p}+\Delta\mathbf{p} \tag{18}$$

with true parameter vector $\mathbf{p}$ and bias term $\Delta\mathbf{p}$. It is easy to see that $\mathbf{R}_{\phi w}$ is nonzero if $w(k)$ and $\boldsymbol{\phi}(k)$ are correlated. Therefore, the bias term does not vanish and distorts the estimation result. Due to the low pass filtering behavior of the PMFs, the LS estimator is still usable in the presence of low measurement noise and relatively large filter time-constants. For some applications, e.g. failure monitoring, it is necessary though to choose small filter time-constants. Therefore, an unbiased estimate is desired.

For its estimation, a further polynomial function $f(s)$ is inserted in the system input to obtain information about the estimation bias [12]. Estimating the bias then amounts to

1) Introduce a known function $f(s)=\prod_{i=1}^{n}(s-z_i)$ with distinct stable zeros z_i in the system input to obtain an extended model

$$y^{(n)}(t)+\sum_{i=0}^{n-1}a_i y^{(i)}(t)=\sum_{j=0}^{n+m_d}b_j^e u^{e(j)}(t) \tag{19}$$

with Laplace transform

$$a(s)Y(s)=b^e(s)U^e(s),\quad U^e(s)=\frac{U(s)}{f(s)} \tag{20}$$

with $a(s)=\sum_{i=0}^{n-1}a_i s^i$, $b^e(s)=\sum_{j=0}^{m_d+n}b_j^e s^{m_d+n-j}$, and the extended parameter vector

$$\mathbf{p}^e=\left(a_0,\cdots,a_{n-1},b_0^e,\cdots,b_{n+m_d}^e\right)^{\mathrm{T}} \tag{21}$$

and filtered input signal $U^e(s)$ where values of the extended systems are denoted by the superscript ‘e’.

2) Estimate the extended biased parameter vector $\hat{\mathbf{p}}_B^e$ using PMF and LS algorithm.
3) Estimate the bias term in the extended model and compute the bias-compensated parameter vector $\hat{\mathbf{p}}_{BC}^e$.
4) Transform parameters to obtain the bias-compensated parameter vector $\hat{\mathbf{p}}_{BC}$ of the original system.

In this context, the additional difficulty is to estimate $\mathbf{R}_{\phi^e w}$. Under the assumption that the noise is uncorrelated to the system input, we have

$$\mathbf{R}_{\phi^e w}=\begin{pmatrix}\mathbf{R}_{yw}^{\mathrm{T}} & \mathbf{R}_{u^e w}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}=\begin{pmatrix}\mathbf{I}_n\\ \hline \mathbf{0}\end{pmatrix}\mathbf{R}_{yw}=\mathbf{G}\mathbf{R}_{yw} \tag{22}$$

due to the input output separation in (13). Since z_i are zeros of $f(s)$, we have that

$$b^e(z_i)=b_0^e z_i^{n+m_d}+b_1^e z_i^{n+m_d-1}+\cdots+b_{n+m_d}^e=0 \tag{23}$$

is satisfied. From (21) and (23) we obtain

$$\mathbf{Z}^{\mathrm{T}}\mathbf{p}^e(N)=\mathbf{0} \tag{24}$$

with the $(2n+m_d+1)\times n$-dimensional coefficient matrix

$$\mathbf{Z}^{\mathrm{T}}=\left(\mathbf{0}_{n\times n}\left|\begin{matrix}z_1^{n+m_d} & \cdots & z_1 & 1\\ z_2^{n+m_d} & \cdots & z_2 & 1\\ \vdots & & \vdots & \vdots\\ z_n^{n+m_d} & \cdots & z_n & 1\end{matrix}\right.\right). \tag{25}$$

In case of distorted parameter estimates (24) reads

$$\mathbf{Z}^{\mathrm{T}}\hat{\mathbf{p}}_B^e(N)=\boldsymbol{\varepsilon} \tag{26}$$

with equation error $\boldsymbol{\varepsilon}$. Inserting (17) and (22) for the augmented system in (26) results in

$$\boldsymbol{\varepsilon}=\mathbf{Z}^{\mathrm{T}}\mathbf{p}^e(N)+\mathbf{Z}^{\mathrm{T}}\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\mathbf{R}_{yw}(N) \tag{27}$$

and from (24) we obtain

$$\boldsymbol{\varepsilon}=\mathbf{Z}^{\mathrm{T}}\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\mathbf{R}_{yw}(N). \tag{28}$$

For sufficient excitation of the system and distinct stable zeros of $f(s)$ to ensure regularity of $\mathbf{Z}^{\mathrm{T}}\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}$ from (26) and (28) we get the estimate

$$\hat{\mathbf{R}}_{yw}(N)=\left(\mathbf{Z}^{\mathrm{T}}\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\right)^{-1}\mathbf{Z}^{\mathrm{T}}\hat{\mathbf{p}}_B^e(N). \tag{29}$$

Consequently, using (18), (22) and (29) an estimate for the bias is given by

$$\Delta\hat{\mathbf{p}}^e(N)=\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\mathbf{R}_{yw}(N) \tag{30}$$

and thus

$$\Delta\hat{\mathbf{p}}^e(N)=\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\left(\mathbf{Z}^{\mathrm{T}}\mathbf{R}_{\phi^e\phi^e}^{-1}(N)\mathbf{G}\right)^{-1}\mathbf{Z}^{\mathrm{T}}\hat{\mathbf{p}}_B^e(N) \tag{31}$$

which leads to the bias-compensated parameter vector

$$\hat{\mathbf{p}}_{BC}^e(N)=\hat{\mathbf{p}}_B^e(N)-\Delta\hat{\mathbf{p}}^e(N). \tag{32}$$

Finally, the bias-compensated parameter vector has to be transformed back again using simple comparison of coefficients from $b^e(s)=b(s)f(s)$ to obtain $\hat{\mathbf{p}}_{BC}$.

In [13] a discrete-time BCRLS algorithm using a prefiltered input is derived. Thus, it is also possible to obtain a recursive algorithm, based on the offline approach presented above. The recursive algorithm with an exponential forgetting term can be written as

$$\begin{aligned}\boldsymbol{\Phi}(k)&=\frac{\mathbf{P}(k-1)\boldsymbol{\phi}^e(k)}{\lambda_{est}+\boldsymbol{\phi}^{e\mathrm{T}}(k)\mathbf{P}(k-1)\boldsymbol{\phi}^e(k)}\\ \mathbf{P}(k)&=\frac{1}{\lambda_{est}}\left(\mathbf{P}(k-1)-\boldsymbol{\Phi}(k)\boldsymbol{\phi}^{e\mathrm{T}}(k)\mathbf{P}(k-1)\right)\\ \mathbf{e}(k)&=\left(y_{PMF}^{(n)}(k)-\boldsymbol{\phi}^{e\mathrm{T}}(k)\mathbf{p}_B^e(k-1)\right)\\ \mathbf{p}_B^e(k)&=\mathbf{p}_B^e(k-1)+\mathbf{P}(k)\boldsymbol{\phi}^e(k)\mathbf{e}(k)\\ \Delta\mathbf{p}^e(k)&=\mathbf{P}(k)\mathbf{G}\left(\mathbf{H}^{\mathrm{T}}\mathbf{P}(k)\mathbf{G}\right)^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{p}_B^e(k)\\ \mathbf{p}_{BC}^e(k)&=\mathbf{p}_B(k)-\Delta\mathbf{p}^e(k)\end{aligned} \tag{33}$$

with $(2n+m_d+1)\times n$-dimensional coefficient matrix $\mathbf{H}$

$$\mathbf{H}^{\mathrm{T}}=\left(\mathbf{0}_{n\times n}\left|\begin{matrix}J_{n+m_d}(\gamma_1) & \cdots & J_0(\gamma_1)\\ J_{n+m_d}(\gamma_2) & \cdots & J_0(\gamma_2)\\ \vdots & & \vdots\\ J_{n+m_d}(\gamma_n) & \cdots & J_0(\gamma_n)\end{matrix}\right.\right). \tag{34}$$

and the calculation rule

$$J_i(k)=(k-1)^i\left(\frac{T}{2}(1+k)\right)^{n+m_d-i} \tag{35}$$

where

$$\gamma_i = e^{z_i T},\ i = 1, \ldots, n \tag{36}$$

describes the zeros of $f(s)$, again discretized by bilinear transformation with sampling time T.

IV. Experiment

For process and failure monitoring tasks it is necessary to observe specific system parameters persistently and to detect deviations from nominal parameters immediately. So as to detect rapid changes in parameters, small filter time constants of the PMFs are necessary. In doing so, it may be expected that rapidly varying measurement noise cannot be suppressed well. This implies that the resulting colored noise may have a significant negative impact on the parameter identification. Moreover, any incoming continuous-time signal is sampled, e.g. by an analog-to-digital converter or an encoder, to produce a discrete-time sequence. Thus, a quantization error will be generated whose size depends on the corresponding sensor resolution. Literature contains numerous studies describing the modeling of such quantization error. It is common practice to describe quantization errors as additive white noise, see [16], [17]. In this section, we study the identification results of RLS and BCRLS with respect to varying encoder resolutions and input signals.

A simple cart system will serve as a reference experiment. The cart with mass m_1 is driven by a dc-drive with fixed external excitation. In the following, we assume a very small electrical time constant ($\tau_{el} = \frac{L_A}{R_A}$ with armature inductance L_A and resistance R_A) compared to the mechanical one. By means of model reduction, i.e. $L_A \to 0$, we introduce equivalent parameters (38) incorporating the whole drive-train (drive constant k_m, inertia J_G, transmission ratio n_i, gear pinion radius r, and mechanical damping d_1) into the mathematical model of the cart. The cart position is measured by a 12 bit encoder with an equivalent resolution of $22.75\,\mu\mathrm{m}$.

The slightly nonlinear system equations read[4]

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \frac{\beta_{DC}}{\tilde{m}_1} u_A - \frac{\tilde{d}_1}{\tilde{m}_1} v - F_C sign(v) \end{pmatrix} \tag{37}$$

with input voltage u_A and the system states: cart position x and the corresponding velocity v. Note that we abbreviated:

$$\begin{aligned} \tilde{m}_1 &= m_1 + J_G \left(\frac{n_i}{r}\right)^2 \\ \tilde{d}_1 &= d_1 + \frac{n_i^2 k_m^2}{r^2 R_A} \end{aligned} \tag{38}$$

We assume the equivalent cart mass $\tilde{m}_1$, the equivalent linear friction coefficient $\tilde{d}_1$ and the Coulomb friction F_C to be unknown. We obtained a machine constant $\beta_{DC} = 2\,\frac{\mathrm{N}}{\mathrm{V}}$, from dedicated measurements. The sampling time is set to $T = 10\,\mathrm{ms}$ for all measurements, see Fig. 3, Fig. 4 and Fig. 5. Note that a different sampling time $T = 50\,\mathrm{ms}$ is used for DOE in order to be able to solve the optimization problem within reasonable effort.

[4] Equation (37) is not fully included in system class (1), but we assume to measure the sign of the velocity.

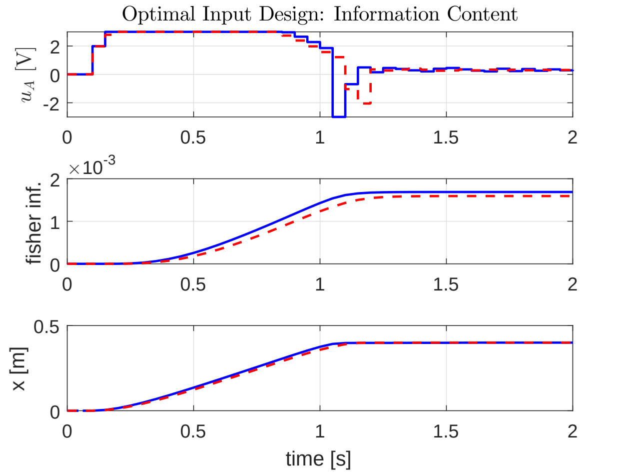

Fig. 2. Impacts of bad initial estimation of plant parameters on optimal input signal. blue ... input design with nominal values, red dashed ... input design with bad initial value of $\tilde{m}_1$.

A. Optimal input design

The optimization problem (5) for optimal input design is exposed in detail for this example. We assume that the system states initially are all zero and that we have a unidirectional movement $(v > 0)$. Thus, we obtain the system

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ \frac{\beta_{DC}}{\tilde{m}_1} u_A - \frac{\tilde{d}_1}{\tilde{m}_1} v - F_C \end{pmatrix} \tag{39}$$

which is of the form (1). After some movement the cart shall be located at $x_e = 0.4\,\mathrm{m}$ at $t_e = 2\,\mathrm{s}$. Furthermore, there are constraints $u_A \in [-3\,\mathrm{V}, 3\,\mathrm{V}]$ and $x \in [0, x_e]$ concerning the input signal and system states, as common in engineering tasks. The cart position x is the measured output y.

The Fisher matrix may be detailed as per

$$\begin{aligned} \mathbf{F}(\mathbf{u}, \mathbf{p}) &= \sum_{k=1}^{N} \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k}^{\mathrm{T}} \left(\sigma^2\right)^{-1} \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k} \\ \left.\frac{\partial y}{\partial \mathbf{p}}\right|_{\mathbf{p},k} &= \left.\frac{\partial x}{\partial \tilde{m}_1}\right|_{\mathbf{p},k} \end{aligned} \tag{40}$$

for best excitation of the unknown parameter $\tilde{m}_1$.

Recall the trade-offs of a bad initial estimate of the plant parameters, as mentioned in the Remark of Section II. The simulations reveal the consequences on the generated input signal, see Fig. 2. We set a deviation on the initial cart mass $\tilde{m}_1$ of approximately 20 percent of its nominal value. As expected, the impact on the information content of the generated input signal is low. However, the desired terminal conditions are not met exactly.

B. Parameter identification

Following Section III-A the convolution of the signal with the impulse response of a stable linear filter can be used to obtain a purely algebraic equation for parameter identification. Equation (37) can be extended, rearranged and transformed to an identification equation of form (12)

$$\bar{\mathbf{y}}(k) = \bar{\boldsymbol{\phi}}^{\mathrm{T}}(k)\,\mathbf{p}^e \tag{41}$$

with

$$\begin{aligned}\bar{\boldsymbol{\phi}}^{\mathrm{T}} =& \Big(\alpha_0\left(x - \frac{\alpha_1}{\alpha_0} g_{f_1} * x - g_{f_0} * x\right), g_{f_1} * x, g_{f_0} * sign(v),\\ & - g_{f_1} * u^e, -\alpha_0\left(u^e - \frac{\alpha_1}{\alpha_0} g_{f_1} * u^e - g_{f_0} * u^e\right)\Big)\\ \bar{\mathbf{y}} =& g_{f_0} * \beta u_A\\ \mathbf{p}^e =& \left(\tilde{m}_1, \tilde{d}_1, F_C, b_0^e, b_1^e\right)^{\mathrm{T}}\end{aligned} \tag{42}$$

the convolution operator '*' and the stable linear filters

$$\begin{aligned} F_0(s) &= \frac{\alpha_0}{s^2 + \alpha_1 s + \alpha_0}, & g_{f_0}(t) &= \mathcal{L}^{-1}\{F_0(s)\}\\ F_1(s) &= sF_0(s), & g_{f_1}(t) &= \mathcal{L}^{-1}\{F_1(s)\}\end{aligned} \tag{43}$$

with free coefficients $\alpha_i > 0$. Note that the sampling index k in (42) has dropped for brevity. Some attention has to be paid to the static friction term. We assume a small number of directional changes such that the Laplace transformation exists. The velocity v in the $sign$-function is approximated by the backward difference quotient and the $sign$-function is approximated by a sum of positive and negative discretized unit steps which only causes a small error as long as there is no chattering around zero velocity. Note that the error vanishes over time with arbitrary filter time constants.

C. Identification results

Due to small filter time constants, the suppression of measurement noise is minor. This in turn leads to an estimation bias using RLS algorithm. As mentioned before, reducing encoder resolution leads to further increased estimation bias. In Fig. 3 the encoder resolution varies from 12 bit (4096 inc) to 9 bit (512 inc) and causes an increased estimation error with RLS algorithm. The results, using BCRLS remain similar subject to changing resolution.

To detect rapid changes in parameters, the least-squares estimation algorithm is extended by an exponential forgetting factor λ_{est}. As expected, varying encoder resolutions lead to biased estimation results using RLS, while BCRLS estimation results again remain similar, see Fig. 4. The oscillations visible for the estimated mass $\tilde{m}_1$ during the first second in Fig. 3 and Fig. 4 are transient effects, caused by setting up the stable filters F_0, F_1. The oscillations within periods of constant mass, e.g. at $t = 3\,\mathrm{s}$, are related to narrow peaks of the input signal that are induced by the position control.

The effect of an optimal input signal on the estimation result is studied with a laboratory experiment, see Fig. 5. The optimal input signal is generated using (5) and (40) and constraints concerning the input signal and system states from Section IV-A. For comparison, a standard pulsed input signal is used. The generated input signal contains higher Fisher information and results in a faster convergence even for standard RLS. To eliminate the remaining stationary

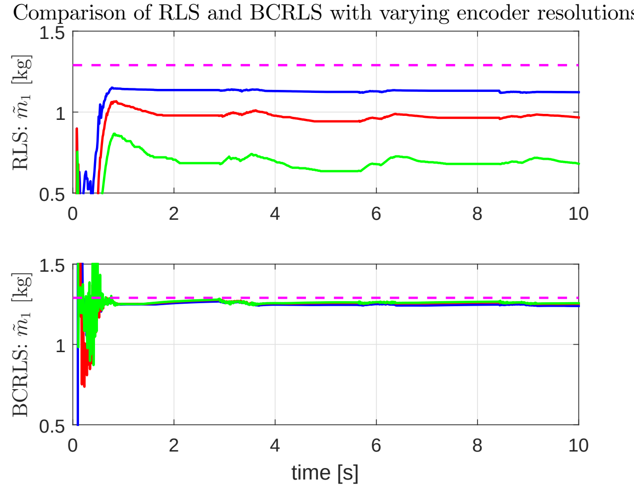

Fig. 3. Comparison of estimated load with varying resolution of encoder. blue ... 4096 inc, red ... 1024 inc, green ... 512 inc, magenta ... nominal value

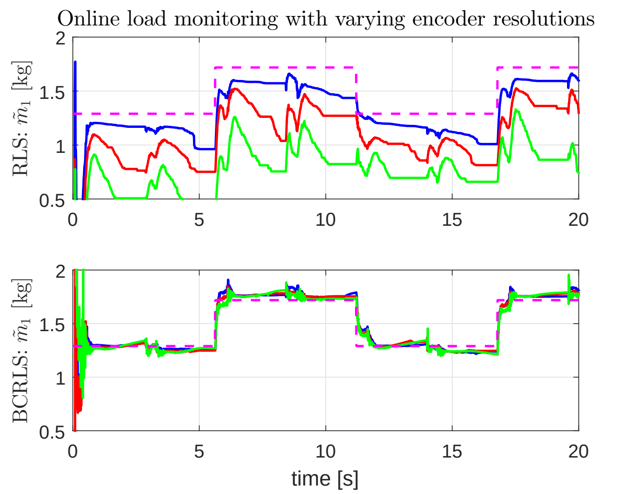

Fig. 4. Online load monitoring with exponential forgetting $\lambda_{est} = 0.99$ for different encoder resolutions. blue ... 4096 inc, red ... 1024 inc, green ... 512 inc, magenta ... nominal value

estimation error and achieve results similar to Fig. 3 and Fig. 4, bias-compensating methods as for example BCRLS are necessary.

V. Conclusion

We have compared the identification results using two continuous-time parameter identification techniques, RLS and BCRLS, with particular attention to online failure monitoring and sensor resolution effects. Obviously, both algorithms benefit from the low-pass behavior of the PMF approach which ensures a remarkable decrease of noise. The BCRLS algorithm leads to superior results when the noise-to-signal ratio is high and the PMF filter time constants are small. In addition, the BCRLS algorithm is able to handle coarser quantized signals without losing identification accuracy. Furthermore, we have generated an optimization based excitation signal for the sufficient excitation of unknown

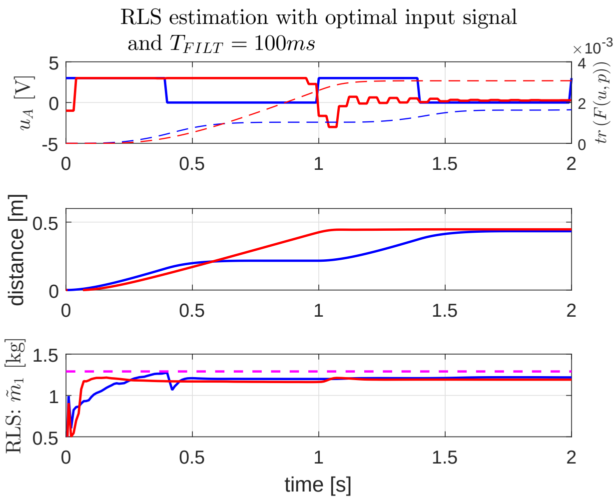

Fig. 5. Online load monitoring using RLS without bias-compensation method and filter time constant $T_{FILT} = 100\,\mathrm{ms}$ with standard input signal and optimal input signal. blue ... standard input, red ... DOE, magenta ... nominal value

plant parameters, already resulting in an improved parameter identification that is independent of the identification method. An accurate plant identification thus is rendered possible by combining optimal input design and bias-compensating identification. This then is in full compliance with signal and state constraints when using low-end sensors.

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